UNIT – V
STATE SPACE ANALYSIS

Topics: Concepts of state - state variables and state model - state space representation of transfer function: Controllable Canonical Form - Observable Canonical Form - Diagonal Canonical Form - diagonalization using linear transformation - solving the time invariant state equations State Transition Matrix and its properties- concepts of controllability and observability.

INTRODUCTION

The concept of modelling, analysis and design of control systems discussed, so far was based on their transfer functions which suffer from some drawbacks as stated below.
(i) The transfer function is defi ned only under zero initial conditions.
(ii) It is only applicable to linear time-invariant systems and generally restricted to single input single output (SISO) systems.
(iii) It gives output for a certain input and provides no information about the internal state of the system.
To overcome these drawbacks in the transfer function, a more generalised and powerful technique of state variable approach in the time domain was developed. The state variable method of modelling, analysis and design is applicable to linear and non-linear, time-invariant or time varying multi-input multi-output (MIMO) systems. The placement of closed-loop poles for improvement of system performance can be done with state feedback.

CONCEPTS OF STATE, STATE VARIABLES AND STATE MODEL

State: state gives the future behaviour of the system based on the present i/p and past history of the system

→ The past history of the system is described by the state variables.

The state of a dynamic system is the smallest set of variables (called state variables) such that the knowledge of these variables at $t = t_0$, together with the knowledge of the inputs for $t \geq t_0$, completely determine the behaviour of the systems for any time $t \geq t_0$

State Variables:

The state variables of a dynamic system are the smallest set of variables that determines the state of the dynamic system.

State Vector:

If n state variables are needed to completely describe the behaviour of a given system, then these n state variables can be considered as the n components of a vector $x(t)$. Such a vector is called a state vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

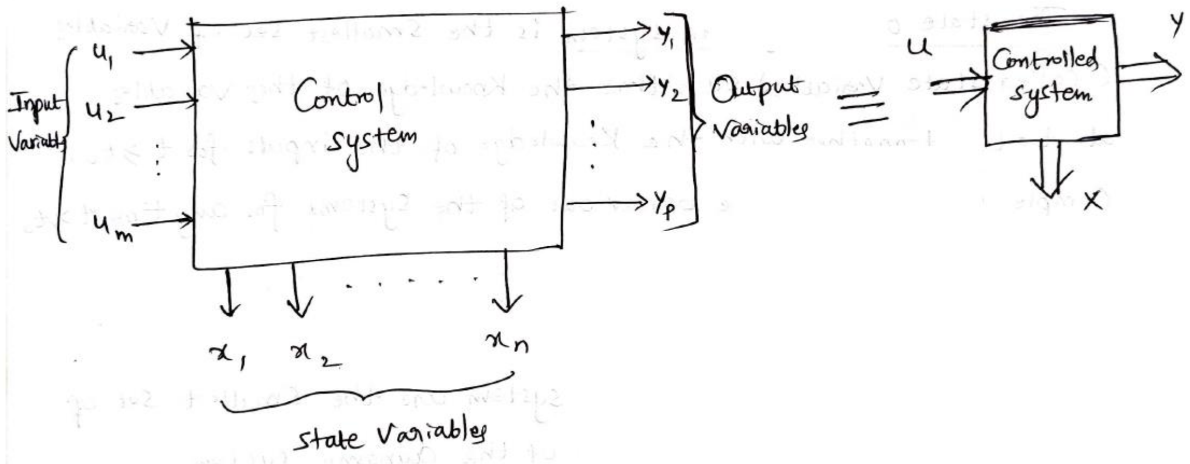
State space:

The n dimensional space whose coordinate axes consist of the x_1 axis, x_2 axis, ..., x_n axis, where $x_1, x_2, \ldots, x_n$ are state variables is called the state space.

STATE SPACE EQUATIONS (OR) STATE MODEL (OR) STATE EQUATIONS

The state-space representation of a given system consists of two equations: (i) state equation

(ii) Output equation.



$x_1(t)$, $x_2(t)$, $x_3(t)$ .... $x_n(t)$ — state variables

$u_1(t)$, $u_2(t)$ .... $u_m(t)$ — input variables

$y_1(t)$, $y_2(t)$ .... $y_p(t)$ ← Output variables

$$u(t) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}_{m\times 1} = \text{Input Vector}$$

$$y(t) = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}_{p\times 1} = \text{O/p vector}$$

$$X(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \text{state vector}$$

The state variable representation of a system can be arranged in the form of n-first order DE's

$$\frac{dx_1}{dt} = \dot{x}_1 = f_1(x_1, x_2, \ldots x_n;\ u_1, u_2, \ldots, u_m)$$

$$\frac{dx_2}{dt} = \dot{x}_2 = f_2(x_1, x_2, \ldots x_n;\ u_1, u_2, \ldots, u_m)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\frac{dx_n}{dt} = \dot{x}_n = f_n(x_1, x_2, \ldots x_n;\ u_1, u_2, \ldots, u_m)$$

The state equations of a LTI system are set of first-order DE's, where each first derivative of the state variable is a linear combination of system states and inputs, i.e.,

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1m}u_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots + b_{2m}u_m.$$

$$\vdots$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{nm}u_m.$$

Similarly, The output variables at time t are linear combination of the input and state variables at time t, i.e.,

$$y_1(t) = c_{11}x_1(t) + c_{12}x_2(t) + \cdots + c_{1n}x_n(t) + d_{11}u_1(t) + d_{12}u_2(t) + \cdots + d_{1m}u_m(t)$$

$$\vdots$$

$$y_p(t) = c_{p1}x_1(t) + c_{p2}x_2(t) + \cdots + c_{pn}x_n(t) + d_{p1}u_1(t) + d_{p2}u_2(t) + \cdots + d_{pm}u_m(t).$$

The set of state equations and output equations of the above state model may be written in a vector-matrix form as given below.

$$\dot{X}(t) = A\,x(t) + B\,U(t)$$

$$Y(t) = C\,x(t) + D\,U(t)$$

Where A = $n \times n$ matrix, $B = n \times 1$ matrix

C = $1 \times n$ matrix, D = constant

and $U(t)$ = single scalar input variable

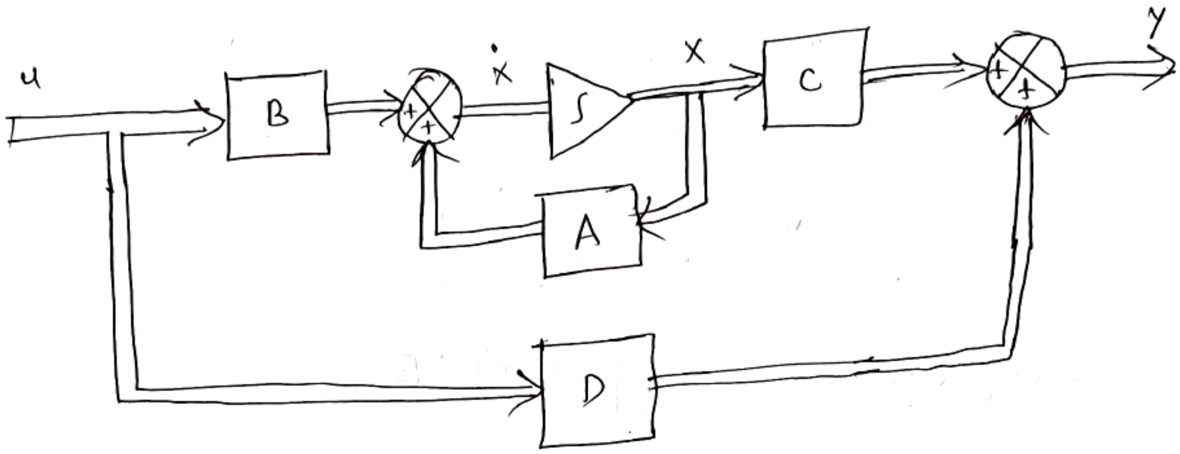
A = Evolution matrix $\Rightarrow n \times n$

B = Control matrix $\Rightarrow n \times m$

C = Observation matrix $\Rightarrow p \times n$

D = Transmission matrix $\Rightarrow p \times m$

The block diagram of state model of linear MIMO system is shown in the fig.



DERIVATION OF TRANSFER FUNCTION FROM STATE MODEL

Let us consider a single-input single-output system the transfer function of which is given by

$$\frac{Y(s)}{U(s)} = G(s)$$

The state model of the above system may be given by the following equations.

$$\dot{x} = Ax + Bu$$
$$y = Cx + dU$$

where x is the state vector, u is the input and y is the output and all are functions of time t.

The Laplace transform given by

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$\text{and} \quad Y(s) = CX(s) + dU(s)$$

As transfer function is defined with zero initial condition, putting $x(0) = 0$

$$sX(s) = AX(s) + BU(s)$$
$$sX(s) - AX(s) = BU(s)$$
$$(sI - A)\, X(s) = BU(s)$$

Pre-multiplying both sides by $(sI - A)^{-1}$, we get,

$$X(s) = (sI - A) -1BU(s)$$

Putting the value of X(s)

$$Y(s) = [C(sI - A)^{-1}B + d]U(s)$$

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + d = \frac{C[adj(sI - A)]B}{|(sI - A)|} + d$$

PROBLEMS

1) Obtain the transfer function of a system described by the following state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

SOL:

Transfer function $= C\,[sI - A]^{-1} B$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{(s+2)^2 - 1} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{s^2+4s+3} \begin{bmatrix} s+2 \\ 1 \end{bmatrix}$$

$$T/F = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{s+2}{s^2+4s+3} \\ \frac{1}{s^2+4s+3} \end{bmatrix}$$

$$\boxed{T/F = \frac{1}{s^2+4s+3}}$$

2) Obtain the transfer function of the system

$$\dot{X} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & -2 & -3 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$Y = [1 \quad 0 \quad 1] X$$

SOL:

$$(sI - A) = \begin{bmatrix} s+1 & 0 & 1 \\ 0 & s+1 & -1 \\ -1 & 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s^3 + 5s^2 + 10s + 6} \begin{bmatrix} s^2+4s+5 & 2 & -(s+1) \\ 1 & s^2+4s+4 & s+1 \\ s+1 & -2(s+1) & (s+1)^2 \end{bmatrix}$$

The transfer function is given by,

$$T(s) = \frac{1}{\Delta(s)} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} s^2+4s+5 & 2 & -(s+1) \\ 1 & s^2+4s+4 & s+1 \\ s+1 & -2(s+1) & (s+1)^2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Where $\Delta(s) = s^3 + 5s^2 + 10s + 6$

$$T(s) = \frac{1}{\Delta(s)} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1-s \\ s^2+5s+5 \\ (s+1)(s-1) \end{bmatrix}$$

$$= \frac{s(s-1)}{s^3+5s^2+10s+6}$$

3) Obtain the transfer function of the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

SOL:

The given system may be written as

$$\dot{x} = \boldsymbol{Ax} + \boldsymbol{Bu}$$

$$y = \boldsymbol{Cx} + du$$

$$A = \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$d = 0$$

$$(s\boldsymbol{I} - \boldsymbol{A}) = \begin{bmatrix} s+4 & 1 \\ -3 & s+1 \end{bmatrix}$$

Thus the adjoint of matrix $(s\boldsymbol{I} - \boldsymbol{A})$ is given by

$$\text{Adj} \quad (s\boldsymbol{I} - \boldsymbol{A}) = \begin{bmatrix} s+1 & -1 \\ 3 & s+4 \end{bmatrix}$$

Also,

$$|s\boldsymbol{I} - \boldsymbol{A}| = (s+4)(s+1) + 3$$
$$= s^2 + 5s + 7$$

We may write the transfer function as

$$\frac{Y(s)}{U(s)} = \frac{\boldsymbol{C}[adj(s\boldsymbol{I} - \boldsymbol{A})]\boldsymbol{B}}{|(s\boldsymbol{I} - \boldsymbol{A})|} + d$$

Now, $\boldsymbol{C}$ [adj $(s\boldsymbol{I} - \boldsymbol{A})$] $\boldsymbol{B}$

$$= [1 \quad 0] \begin{bmatrix} s+1 & -1 \\ 3 & s+4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= [1 \quad 0] \begin{bmatrix} s \\ s+7 \end{bmatrix} = s$$

$$\therefore \qquad \frac{Y(s)}{U(s)} = \frac{s}{s^2 + 5s + 7}$$

4) Find the transfer function when

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{and } C = [1 \quad 1]$$

SOL:

$$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} s+2 & -1 \\ 0 & s+3 \end{bmatrix}$$

$$|sI - A| = \begin{vmatrix} s+2 & -1 \\ 0 & s+3 \end{vmatrix} = (s+2)(s+3) \neq 0$$

Therefore, $(sI - A)^{-1}$ exists.

Now

$$(sI - A)^{-1} = \frac{Adj(sI - A)}{|sI - A|} = \frac{\begin{bmatrix} s+3 & -1 \\ 0 & s+3 \end{bmatrix}}{(s+2)(s+3)}$$

$$C(sI - A)^{-1}B = [1 \quad 1] \frac{\begin{bmatrix} s+3 & -1 \\ 0 & s+2 \end{bmatrix}}{(s+2)(s+3)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{[1 \quad 1]\begin{bmatrix} 1 \\ s+2 \end{bmatrix}}{(s+2)(s+3)} = \frac{(s+3)}{(s+2)\ (s+3)} = \frac{1}{s+2}$$

STATE SPACE REPRESENTATION FOR ELECTRICAL NETWORKS (PHYSICAL VARIABLE FORM)

PROCEDURE:

(1) Select the state variables as Voltage across capacitor and Current through the inductor.

(2) The no. of state variables is equal to sum of the inductors & capacitors.

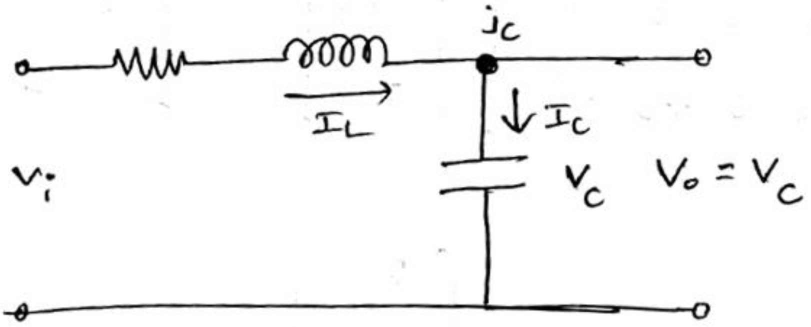
(3) Apply independent KCL and KVL

(4) At capacitor junction, apply KCL
Apply KVL through inductor

(5) The resultant equation should consists state variables, differentiation of state variables, i/p variables and o/p variables.

PROBLEMS:

1) Obtain the state model for the following network



SOL:

KCL at j_c :

$$I_L = I_c = C \cdot \frac{dv_c}{dt}$$

$$\dot{V}_c = \frac{I_L}{C} \longrightarrow (1)$$

KVL through inductor :

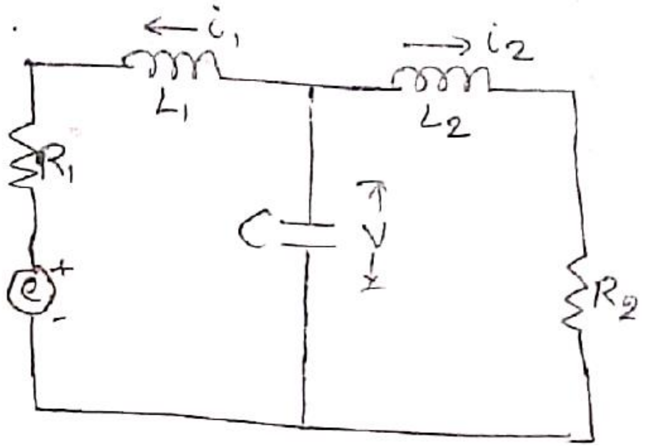
$$V_i = I_L R + L \cdot \frac{dI_L}{dt} + V_C$$

$$\Rightarrow \dot{I}_L = \frac{V_i}{L} - \frac{R}{L} I_L - \frac{V_C}{L} \longrightarrow (2)$$

$$\begin{bmatrix} \dot{V}_C \\ \dot{I}_L \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} V_C \\ I_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} [V_i]$$

$$V_o = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} V_C \\ I_L \end{bmatrix}$$

2) Obtain the state model for the following network



SOL:

Consider the State variables as i_1, i_2, v

Let $x_1(t) = V(t)$

$x_2(t) = i_1(t)$

$x_3(t) = i_2(t)$

The differential eq.s for the given circuit are

$$i_1 + i_2 + C\frac{dv}{dt} = 0 \longrightarrow (1)$$

$$L_1 \frac{di_1}{dt} + R_1 i_1 + e - V = 0 \longrightarrow (2)$$

$$L_2 \frac{di_2}{dt} + R_2 i_2 - V = 0 \longrightarrow (3)$$

From eq ① $\Rightarrow$ $\dot{x}_1 = -\frac{1}{C} i_1 - \frac{1}{C} i_2 = \frac{dv}{dt}$

From eq. ② $\Rightarrow$ $\dot{x}_2 = \frac{di_1}{dt} = \frac{1}{L_1} V - \frac{R_1}{L_1} i_1 - \frac{1}{L_1} e$

From eq ③ $\Rightarrow$ $\dot{x}_3 = \frac{di_2}{dt} = \frac{1}{L_2} V - \frac{R_2}{L_2} i_2$

The above eqs can be redrawn in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -1/C & -1/C \\ 1/L_1 & -R_1/L_1 & 0 \\ 1/L_2 & 0 & -R_2/L_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/L_1 \\ 0 \end{bmatrix} u$$

where $u = e(t)$ ——— ④

Assume voltage across R_2 and current through R_2 are the o/p variables.

$Y_1 = i_2 R_2 = x_3 R_2$

$Y_2 = i_2 = x_3$

The above eqs are redrawn as

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & R_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longleftarrow ⑤$$

The eqs ④, ⑤ are state model or state space representation of the system.

STATE SPACE REPRESENTATION FOR DIFFERENTIAL EQUATIONS

PROBLEMS:

1) Construct the state model for a system characterized by the differential equation

$$\ddot{y} + 5\dot{y} + 6y = u$$

SOL:

Let $y = x_1$

$$\dot{y} = \dot{x}_1 = x_2$$

$$\ddot{y} = \dot{x}_2 = x_3$$

$$\dot{x}_2 = u - 5x_2 - 6x_1$$

$$= u - 6x_1 - 5x_2$$

The state model is given as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\dot{x} = Ax + Bu$$

and o/p eq $y = Cx$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2) Construct the state model for a system characterized by the differential equation

$$\dddot{y} + 6\ddot{y} + 10\dot{y} + 5y = u$$

SOL:

$x_1 = y$	i.e.	$y = x_1$
$x_2 = \dot{y} = \dot{x}_1$		$\dot{x}_1 = x_2$
$x_3 = \ddot{y} = \dot{x}_2$		$\dot{x}_2 = x_3$
$\dddot{y} = -6\ddot{y} - 10\dot{y} - 5y + u$		$\dot{x}_3 = -5x_1 - 10x_2 - 6x_3 + u$

Therefore, the state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -10 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

3) Construct the state model for a system characterized by the differential equation

$$\dddot{y} + 5\ddot{y} + 6\dot{y} + 7y = 10u$$

SOL:

No. of state variables required = 3

Let $y = x_1$

$x_2 = \dot{y} = \dot{x}_1$

$x_3 = \ddot{y} = \dot{x}_2$

$\dddot{y} = \dot{x}_3$

Substitute all in the given equation

$\dot{x}_3 + 5x_3 + 6x_2 + 7x_1 = 10u$

$\dot{x}_3 = 10u - 7x_1 - 6x_2 - 5x_3$

State Model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} [u]$$

$$[y] = Cx + Du$$

$$[y] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

CONTROLLABLE CANONICAL FORM

STATE SPACE REPRESENTATION FOR TRANSFER FUNCTION (PHASE VARIABLE FORM)

PROBLEMS

1) Obtain the state model for the following transfer function

$$T(S) = \frac{b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

SOL:

Given $$\frac{Y(s)}{U(s)} = \frac{b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

$$\frac{d^3y}{dt^3} + a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b_0 u$$

let $y = x_1$

$\dot{y} = \dot{x}_1 = x_2$

$\ddot{y} = \dot{x}_2 = x_3$

$\dddot{y} = \dot{x}_3$

$$\dddot{y}(t) + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_0 u(t)$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

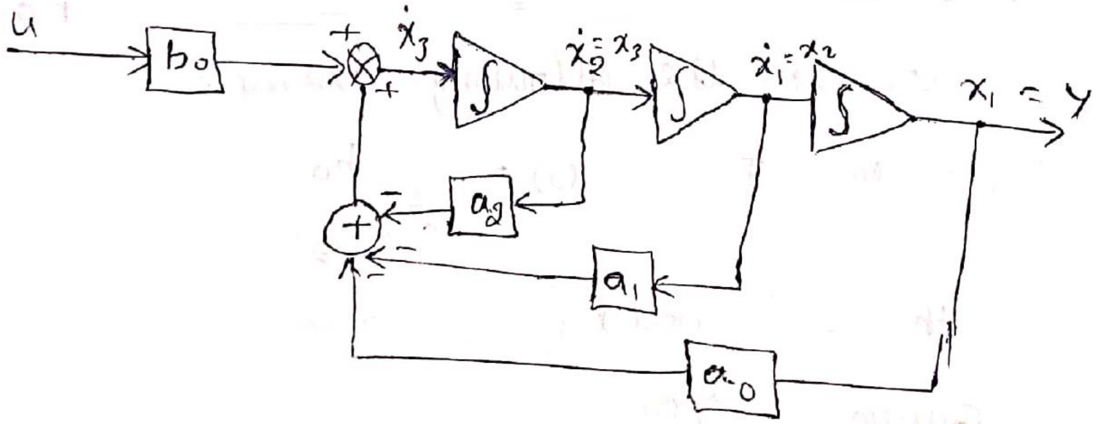
$$\dot{x}_3 = -a_0 x_1 - a_1 x_2 - a_2 x_3 + b_0 u$$

The state model can be redrawn as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The block diagram for the state model is given as



2) Obtain the state model for the following transfer function

$$G(s) = \frac{1}{s^3 + 4s^2 + 3s + 3}$$

SOL:

$$G(s) = \left. \frac{Y(s)}{U(s)} \right|_{\text{initial Conditions} = 0}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^3+4s^2+3s+3}$$

$$\Rightarrow Y(s)\left[s^3+4s^2+3s+3\right] = U(s)$$

Taking Inverse Laplace transform with all initial Conditions zero

$$\dddot{y} + 4\ddot{y} + 3\dot{y} + 3y = u$$

Let $y = x_1$

$$x_2 = \dot{y} = \dot{x}_1$$

$$x_3 = \ddot{y} = \dot{x}_2$$

$$\dddot{y} = \dot{x}_3$$

$$\dot{x}_3 + 4x_3 + 3x_2 + 3x_1 = u$$

$$\dot{x}_3 = u - 3x_1 - 3x_2 - 4x_3$$

State equation in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Output equation

$$y = x_1$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

3)

Obtain the state model of the system whose transfer function is given by $\frac{s^2+7s+2}{s^3+9s^2+26s+24}$

SOL:

$$\frac{Y(s)}{U(s)} = \frac{s^2+7s+2}{s^3+9s^2+26s+24}$$

$$\frac{Y(s)}{U(s)} = \frac{Y(s)}{C(s)} \times \frac{C(s)}{U(s)}$$

$$\frac{Y(s)}{C(s)} = s^2 + 7s + 2 \qquad \text{--------(1)}$$

$$\frac{C(s)}{U(s)} = \frac{1}{s^3+9s^2+26s+24} \qquad \text{-------(2)}$$

Consider equation (2), $\frac{C(s)}{U(s)} = \frac{1}{s^3+9s^2+26s+24}$

Cross-multiplying on both sides,

$[s^3 + 9s^2 + 26s + 24]\, C(s) = U(s)$

$s^3\, C(s) + 9s^2\, C(s) + 26s\, C(s) + 24\, C(s) = U(s)$

Taking inverse Laplace transform,

$$\frac{d^3c(t)}{dt^3} + 9\frac{d^2c(t)}{dt^2} + 26\frac{dc(t)}{dt} + 24\, c(t) = u(t)$$

$\dddot{c}(t) + 9\,\ddot{c}(t) + 26\,\dot{c}(t) + 24\, c(t) = u(t)$

$x_1(t) = c(t)$

$\dot{x}_1(t) = x_2(t) = \dot{c}(t) \qquad$ -----(3)

$\dot{x}_2(t) = x_3(t) = \ddot{c}(t) \qquad$ -----(4)

$\dot{x}_3(t) = \dddot{c}(t)$

$\dddot{c}(t) + 9\ddot{c}(t) + 26\dot{c}(t) + 24c(t) = u(t)$

$\dot{x}_3(t) + 9\, x_3(t) + 26x_2(t) + 24x_1(t) = u(t)$

$\dot{x}_3(t) = -\, 24x_1(t) - 26x_2(t) - 9\, x_3(t) + u(t)$ -----(5)

Putting equations 3, 4 and 5 in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [u]$$

Consider equation (1),

$$\frac{Y(s)}{C(s)} = s^2 + 7s + 2$$

$$Y(s) = [s^2 + 7s + 2]C(s)$$

$$Y(s) = s^2C(s) + 7sC(s) + 2C(s)$$

Taking inverse Laplace transform,

$$y(t) = \ddot{c}(t) + 7\dot{c}(t) + 2c(t)$$

$$y(t) = 2x_1(t) + 7x_2(t) + x_3(t)$$

$$y(t) = [2 \;\; 7 \;\; 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

OBSERVABLE CANONICAL FORM

The state model in controllable canonical form is given by

$$\dot{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

$$y = \boldsymbol{C}\boldsymbol{x} + du$$

The state model in observable canonical form from the controllable canonical form is given by

$$\dot{x} = A^T x(t) + C^T u(t)$$

$$y = B^T x(t)$$

$$\frac{Y(s)}{U(s)} = \frac{b_o s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

The following state-space representation is called an observable canonical form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_o \\ b_{n-1} - a_{n-1} b_o \\ \vdots \\ b_2 - a_2 b_o \\ b_1 - a_1 b_o \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_o u$$

PROBLEM:

1) Obtain the state model in observable canonical form for the following transfer function

$$G(s) = \frac{1}{s^3+4s^2+3s+3}$$

SOL:

State equation in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Output equation

$$y = x_1$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The state model in observable canonical form is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -3 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

DIAGONAL CANONICAL FORM

The General TF of the system is given as

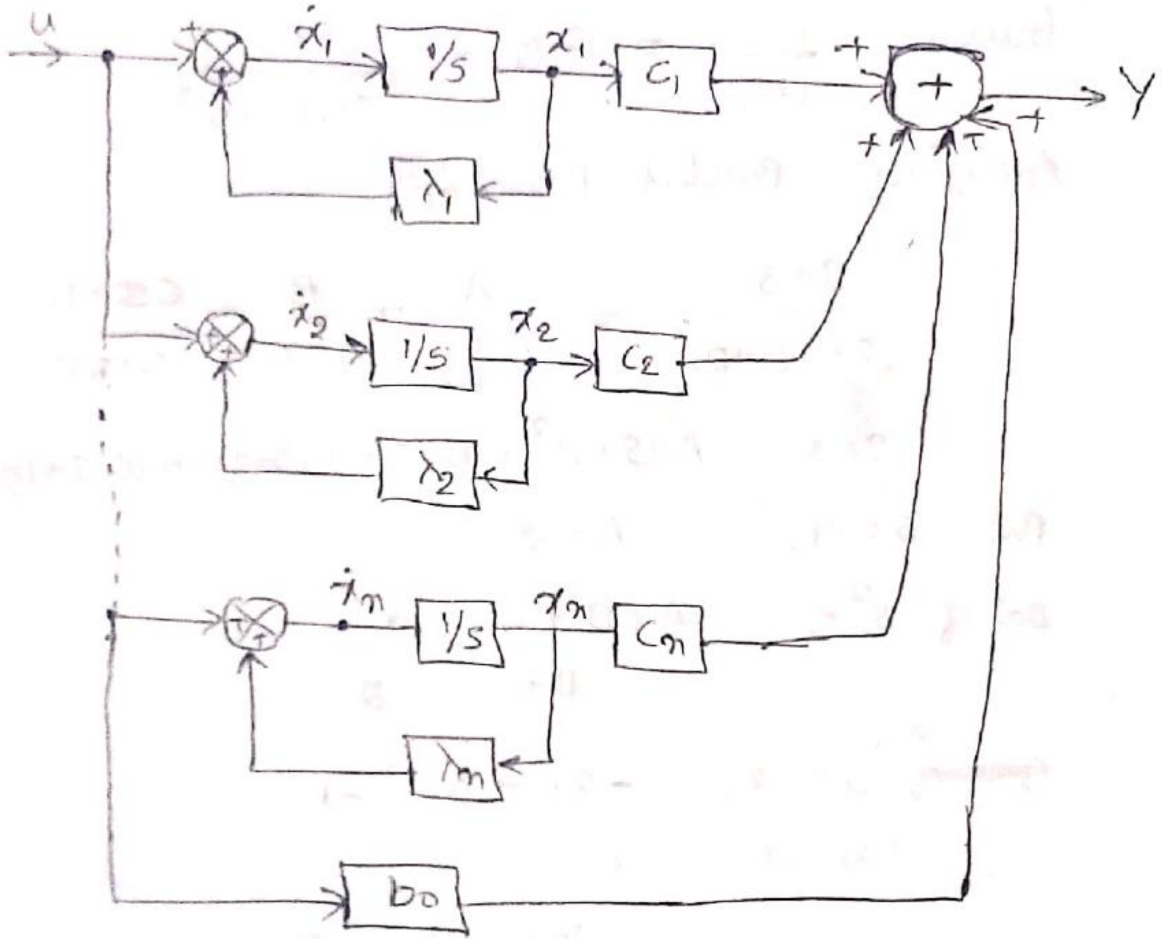
$$\frac{Y(S)}{U(S)} = T(S) = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

The TF can be expanded into partial fractions

$$\frac{Y(S)}{U(S)} = b_0 + \sum_{i=1}^{n} \frac{c_i}{s-\lambda_i}$$

where c_i are residues of the poles at $s = \lambda_i$

The block diagram model for the TF is shown below.



The o/p of the each integrator to be a state variable.
The state eqs are

$$\dot{x}_i = \lambda_i x_i + u \quad \text{where } i = 1, 2, \ldots n \qquad (1)$$

The o/p $y(t)$ is given by

$$y = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n + b_0 u \qquad (2)$$

The eqs (1), (2) are called canonical form of state model of the TF.

This state model can be expressed in vector matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdot & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ & & & & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

PROBLEM

1)

A control system has a TF given by $G(s) = \dfrac{s+3}{(s+1)(s+2)^2}$ obtain the canonical state variable representation.

SOL:

Given $\dfrac{Y(s)}{U(s)} = G(s) = \dfrac{s+3}{(s+1)(s+2)^2}$

Applying Partial Fractions

$$\frac{s+3}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$s+3 = A(s+2)^2 + B(s+1)(s+2) + (C)(s+1)$$

Put $s = -1$, $A = 2$

co. of s^2, $A + B = 0$

$B = -2$

Put

$s = -2$, $-C = +1$

$C = -1$

$$\frac{Y(s)}{U(s)} = \frac{2}{s+1} + \frac{-2}{s+2} + \frac{-1}{(s+2)^2}$$

In this case, $b_0 = 0$

The State model in matrix form are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u$$

& $$y = \begin{bmatrix} 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

2) Obtain the state model in diagonal canonical form for the following transfer function

$$\frac{Y(s)}{U(s)} = \frac{2(s+5)}{(s+2)(s+3)(s+4)}$$

SOL:

$$\frac{Y(s)}{U(s)} = \frac{2(s+5)}{(s+2)(s+3)(s+4)}$$

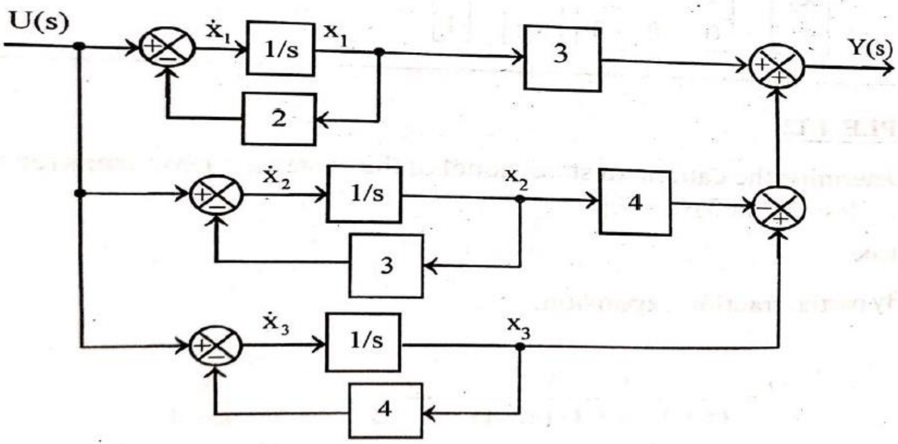
By partial fraction expansion,

$$\frac{Y(s)}{U(s)} = \frac{2(s+5)}{(s+2)(s+3)(s+4)} = \frac{A}{(s+2)} + \frac{B}{(s+3)} + \frac{C}{(s+4)}$$

Solving for A, B and C

A = 3; B = - 4; C = 1

$$\frac{Y(s)}{U(s)} = \frac{2(s+5)}{(s+2)(s+3)(s+4)} = \frac{3}{(s+2)} - \frac{4}{(s+3)} + \frac{1}{(s+4)}$$



The state equations are

$$\dot{x}_1 = -2x_1 + u$$

$$\dot{x}_2 = -3x_2 + u$$

$$\dot{x}_3 = -4x_3 + u$$

The output equation is

$$y = 3x_1 - 4x_2 + x_3$$

The State model is given by,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-2 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}[u]$$

$$Y = \begin{bmatrix}3 & -4 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

DIAGONALIZATION USING LINEAR TRANSFORMATION

The diagonal matrix plays an important role in the matrix algebra. The eigen values and inverse of a diagonal matrix can be very easily obtained just by an inspection.

When matrix A is diagonalised, then the elements along its principle diagonal are the eigen values. The eigen values are the closed loop poles of the system, from which the stability of the system can be analysed.

Consider nth order state model in which matrix A is not diagonal.

$$\dot{x}(t) = A\,x(t) + B\,u(t) \quad \text{——— (1)}$$

$$Y(t) = C\,x(t) + D\,u(t) \quad \text{——— (2)}$$

let $z(t)$ is a new state vector such that the transformation is

$$x(t) = M\,z(t) \quad \text{——— (3)}$$

where M = Modal matrix of A

$$\therefore \dot{x}(t) = M\,\dot{z}(t) \quad \text{——— (4)}$$

Sub. eq. (3) & (4) into eq. (1) & (2)

$$\therefore M\,\dot{z}(t) = A M\,z(t) + B\,u(t) \quad \text{——— (5)}$$

$$Y(t) = C M\,z(t) + D\,u(t)$$

Pre-multiplying eq. (5) by M^{-1} on both sides

Pre-multiplying eq. (5) by M^{-1} on both sides

$$\therefore \; M^{-1}M \, \dot{Z}(t) = M^{-1}AM \, Z(t) + M^{-1}B \, u(t)$$

$$\Rightarrow \dot{Z}(t) = M^{-1}AM \, Z(t) + M^{-1}B \, u(t) \quad \text{—— (6)}$$

$$\& \; Y(t) = CM \, Z(t) + D \, u(t) \quad \text{—— (7)}$$

The eq. (6) & (7) gives the canonical state model and $M^{-1}AM$ is a diagonal matrix and denoted by Λ.

$$\therefore \text{ diagonal matrix}, \; \Lambda = M^{-1}AM.$$

Modal Matrix (M) :- A matrix which is obtained by placing all the eigen vectors together is called a modal matrix or diagonalising matrix M.

$$\therefore M = [M_1 : M_2 : \ldots : M_n]$$

PROBLEM:

1) consider a state model with matrix A is given as

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 4 & 0 & 1 \\ -48 & -34 & -9 \end{bmatrix}$$

Determine (a) characteristic eq. (b) Eigen values (c) Eigen vectors (d) modal matrix (e) diogonal matrix

SOL:

(a) Characteristic eq. :-

The characteristic eq. is $|\lambda I - A| = 0$

$$\Rightarrow \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 2 & 0 \\ 4 & 0 & 1 \\ -48 & -34 & -9 \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda & -2 & 0 \\ -4 & \lambda & -1 \\ 48 & 34 & \lambda+9 \end{vmatrix} = 0 \qquad \text{—— (1)}$$

$$\Rightarrow \lambda(\lambda^2+9\lambda+34) + 2(-4\lambda-36+48) + 0 = 0$$

$$\lambda^3 + 9\lambda^2 + 34\lambda - 8\lambda - 72 + 96 = 0$$

$\therefore$ characteristic eq is $\lambda^3 + 9\lambda^2 + 26\lambda + 24 = 0$

(b) Eigen values :-

The characteristic eq. is $\lambda^3 + 9\lambda^2 + 26\lambda + 24 = 0$

$$\Rightarrow (\lambda+2)(\lambda+3)(\lambda+4) = 0$$

$\therefore$ eigen values of matrix A are, $\lambda_1 = -2$

$\lambda_2 = -3$

$\lambda_3 = -4$

(c) Eigen vectors :- To find eigen vectors, obtain matrix $[\lambda_i I - A]$ for each eigen value.

$\therefore$ For $\lambda_1 = -2$, $[\lambda_i I - A] = \begin{bmatrix} -2 & -2 & 0 \\ -4 & -2 & -1 \\ 48 & 34 & 7 \end{bmatrix}$

$\therefore M_1 = \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{bmatrix}$ = cofactors of row 1

$M_1 = \begin{bmatrix} 20 \\ -20 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ {$\because$ Taken out the common factor}

For $\lambda_2 = -3$, $[\lambda_i I - A] = \begin{bmatrix} -3 & -2 & 0 \\ -4 & -3 & -1 \\ 48 & 34 & 6 \end{bmatrix}$.

$\therefore M_2 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 16 \\ -24 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

For $\lambda_3 = -4$, $[\lambda_i I - A] = \begin{bmatrix} -4 & -2 & 0 \\ -4 & -4 & -1 \\ 48 & 34 & 5 \end{bmatrix}$.

$\therefore M_3 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \begin{bmatrix} 14 \\ -28 \\ 56 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

$\therefore M_1, M_2$ & M_3 are the eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2, \lambda_3$.

(d) Modal Matrix (M) :-

The modal matrix, $M = [M_1 : M_2 : M_3]$

$$\therefore M = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -3 & -2 \\ -2 & 1 & 4 \end{bmatrix}$$

(e) Diagonal matrix ($M^{-1}AM$) :-

$$M^{-1} = \frac{\text{Adj } M}{|M|} = \frac{[\text{cofactors of } M]^T}{|M|}$$

$$\therefore \text{Adj } M = \begin{bmatrix} -10 & 8 & -7 \\ -7 & 6 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} -10 & -7 & -1 \\ 8 & 6 & 1 \\ -7 & -5 & -1 \end{bmatrix}$$

$$|M| = 1(-12+2) - 2(-4-4) + 1(-1-6)$$
$$= -10 + 16 - 7 = -1$$

$$M^{-1} = \frac{\text{Adj } M}{|M|} = \begin{bmatrix} 10 & 7 & 1 \\ -8 & -6 & -1 \\ 7 & 5 & 1 \end{bmatrix}$$

$$AM = \begin{bmatrix} 0 & 2 & 0 \\ 4 & 0 & 1 \\ -48 & -34 & -9 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & -3 & -2 \\ -2 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -6 & -4 \\ 2 & 9 & 8 \\ 4 & -3 & -16 \end{bmatrix}$$

$$\therefore \quad M^{-1}AM = \begin{bmatrix} 10 & 7 & 1 \\ -8 & -6 & -1 \\ 7 & 5 & 1 \end{bmatrix} \begin{bmatrix} -2 & -6 & -4 \\ 2 & 9 & 8 \\ 4 & -3 & -16 \end{bmatrix}$$

diagonal matrix, $\Lambda = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

SOLUTION OF STATE EQUATIONS & STATE TRANSITION MATRIX (STM)

For a time-invariant system, the state equations are divided into two types. They are

i) Homogeneous State equations
ii) Non-Homogeneous State equations

SOLUTION OF HOMOGENEOUS STATE EQUATION

For homogeneous eq., A is constant matrix
$u(t)$ is zero vector.

$$\dot{x} = Ax$$

Taking the Laplace transform of both sides of Equation

$$sX(s) - x(0) = AX(s)$$

$$(sI - A)X(s) = x(0)$$

Premultiplying both sides of this last equation by $(sI - A)^{-1}$, we obtain

$$X(s) = (sI - A)^{-1}x(0)$$

$$X(S) = \phi(S)\, x(0)$$

where $\phi(S) = (SI - A)^{-1}$ = resolvent matrix

$$= \frac{1}{S}\left(I - \frac{A}{S}\right)^{-1}$$

$$= \frac{1}{S}\left[I + \frac{A}{S} + \frac{A^2}{S^2} + \dots\dots\right] \left\{\because (1-x)^{-1} = 1 + x + x^2\right\}$$

$$\phi(S) = \left[\frac{I}{S} + \frac{A}{S^2} + \frac{A^2}{S^2} + \dots\right]$$

Taking inverse Laplace Transform

$$\phi(t) = \left[I + At + \frac{A^2t^2}{2!} + \dots\right]$$

$$= e^{At}$$

$$X(S) = \phi(S)\, x(0)$$

Taking inverse Laplace Transform

$$x(t) = \phi(t)\, x(0)$$

$$x(t) = e^{At}\, x(0)$$

where $\phi(t)$ is called State Transition matrix (STM)

PROPERTIES OF STATE TRANSITION MATRIX (Φ(t))

i) $\phi(t) = e^{At}$

Put $t = 0$

$$\phi(0) = e^0 = I$$

ii) $\phi(t) = e^{At}$

Post multiply by e^{-At} on both sides

$$\phi(t)\, e^{-At} = e^{At} e^{-At} = I$$

Pre multiply by $\phi^{-1}(t)$ on both sides

$$\phi^{-1}(t)\, \phi(t)\, e^{-At} = \phi^{-1}(t)$$

$$\because e^{-At} = \phi(-t)$$

$$\phi^{-1}(t)\, \phi(t)\, \phi(-t) = \phi^{-1}(t)$$

$$\phi(-t) = \phi^{-1}(t)$$

$$\therefore \boxed{\phi^{-1}(t) = \phi(-t)}$$

iii) $$\phi(t_2 - t_1)\, \phi(t_1 - t_0) = e^{A(t_2 - t_1)} e^{A(t_1 - t_0)}$$

$$= e^{A(t_2 - t_0)}$$

$$= \phi(t_2 - t_0)$$

iv) $$[\phi(t)]^k = e^{At} e^{At} e^{At} \ldots\ k \text{ times}$$

$$= e^{Akt}$$

$$= \phi(kt) \quad k = +ve \text{ integer.}$$

v) $$\dot{\phi}(t) = \frac{d}{dt}(e^{At}) = A e^{At}$$

$$\dot{\phi}(t) = A\, \phi(t)$$

SOLUTION OF NON-HOMOGENEOUS STATE EQUATION

For Non-homogeneous eq, $u(t)$ is taken into account.

i.e. $\dot{x}(t) = Ax(t) + Bu(t)$

Taking Laplace Transform

$$S\,X(S) - x(0) = A\,X(S) + B\,U(S)$$

$$(SI - A)\,X(S) = x(0) + B\,U(S)$$

$$X(S) = (SI - A)^{-1}x(0) + (SI - A)^{-1}B\,U(S)$$

Taking inverse Laplace Transform

$$x(t) = LT^{-1}\left[(SI - A)^{-1}x(0) + (SI - A)^{-1}B\,U(S)\right]$$

$$\Rightarrow \quad LT^{-1}\{(SI - A)^{-1}x(0)\} = e^{At}x(0) \qquad \text{(1)}$$

$$= \phi(t)\,x(0)$$

$$\& \quad LT^{-1}\left[(SI - A)^{-1}B\,U(S)\right] = \int_0^t e^{A(t-x)}B\,u(x)\,dx$$

Sub. these values in eq. (1)

$$\therefore \quad x(t) = \underbrace{e^{At}x(0)}_{\text{Homogeneous solution}} + \underbrace{\int_0^t e^{A(t-x)}B\,u(x)\,dx}_{\text{Forced solution.}}$$

NOTE :

$$L^{-1}\{\phi(s)\,B\,U(s)\} = \int_0^t \phi(t-\tau)\,BU(\tau)\,d\tau$$

PROBLEMS

1) Find the homogeneous solution of the system

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} X; \qquad X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

SOL:

The solution of the given system is given by,

$$X(t) = e^{At} X_0$$

Let us compute the state transition matrix e^{At} using Laplace transform method.

$$e^{At} = \mathcal{L}^{-1} [(sI - A)^{-1}]$$

$$(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+3}{(s+1)\ (s+2)} & \frac{1}{(s+1)\ (s+2)} \\ \frac{-2}{(s+1)\ (s+2)} & \frac{s}{(s+1)\ (s+2)} \end{bmatrix}$$

$$e^{At} = \mathcal{L}^{-1} [sI - A]^{-1}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

The homogeneous solution of the state equation is given by,

$$X(t) = e^{At} X_0$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} \\ -2e^{-t} - 2e^{-2t} \end{bmatrix}$$

2) Find the homogeneous solution of the system

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad \text{and} \quad X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

SOL:

From the given model,

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$[s\,I - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$$

$$\text{Adj}\,[s\,I - A] = \begin{bmatrix} s-1 & 1 \\ 0 & s-1 \end{bmatrix}^T = \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}$$

$$|sI - A| = (s-1)^2$$

$$[s\,I - A]^{-1} = \frac{\text{Adj}\,[s\,I - A]}{|s\,I - A|} = \frac{\begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}}{(s-1)^2}$$

$$= \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

$$e^{At} = L^{-1}[s\,I - A]^{-1} = L^{-1}\begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

$$X(t) = e^{At}\,X(0) = \text{zero input response}$$

$$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^t \\ t\,e^t \end{bmatrix}$$

3)

$$\textit{Given}\ \dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(t)$$

Find the unit step response when, $X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

SOL:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\text{Adj } [sI - A] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}^T = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$|sI - A| = \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} = s^2 + 3s + 2 = (s+1)\ (s+2)$$

$$[sI - A]^{-1} = \frac{\text{Adj } [sI - A]}{|sI - A|} = \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\[2ex] \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix} = \phi(s)$$

$$\phi(s) = \begin{bmatrix} \dfrac{2}{s+1} - \dfrac{1}{s+2} & \dfrac{1}{s+1} - \dfrac{1}{s+2} \\[2ex] \dfrac{-2}{s+1} + \dfrac{2}{s+2} & \dfrac{-1}{s+1} + \dfrac{2}{s+2} \end{bmatrix}$$

$$e^{At} = L^{-1}[\phi(s)] = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\[2ex] -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$ZIR = e^{At}X(0) = e^{At}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} + e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} - e^{-t} + 2e^{-2t} \end{bmatrix} = \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix}$$

$$ZSR = L^{-1}\{\phi(s)BU(s)\} \qquad \text{where } U(s) = \frac{1}{s} \text{ due to unit step input}$$

$$= L^{-1}\left\{\begin{bmatrix} \dfrac{(s+3)}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\left[\frac{1}{s}\right]\right\}$$

$$= L^{-1}\left\{\begin{bmatrix} \dfrac{1}{s(s+1)(s+2)} \\ \dfrac{1}{(s+1)(s+2)} \end{bmatrix}\right\} = L^{-1}\begin{bmatrix} \dfrac{0.5}{s}-\dfrac{1}{s+1}+\dfrac{0.5}{s+2} \\ \dfrac{1}{s+1}-\dfrac{1}{s+2} \end{bmatrix}$$

$$= \begin{bmatrix} 0.5-e^{-t}+0.5\,e^{-2t} \\ e^{-t}-e^{-2t} \end{bmatrix} s$$

$$\therefore\ Y(t) = ZIR + ZSR = \begin{bmatrix} 3e^{-t}-2e^{-2t} \\ -3e^{-t}+4e^{-2t} \end{bmatrix} + \begin{bmatrix} 0.5-e^{-t}+0.5e^{-2t} \\ e^{-t}-e^{-2t} \end{bmatrix} = \begin{bmatrix} 0.5+2e^{-t}-1.5e^{-2t} \\ -2e^{-t}+3e^{-2t} \end{bmatrix}$$

4)

A system is given by the following vector-matrix equations.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -4 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u$$

where the initial condition is given by $x(0) = [1 \quad 1]^T$

Determine (a) State transition matrix, (b) Zero input response, (c) Zero state response for $u = 1$, (d) Total response, and (e) Inverse of state transition matrix.

SOL:

The given state equation may be written as

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$$

where, $\boldsymbol{A} = \begin{bmatrix}0 & 1\\ -4 & -5\end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix}0\\1\end{bmatrix}$

(a) State transition matrix $\mathbf{\Phi}(t)$

$$s\boldsymbol{I} - \boldsymbol{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 4 & s+5 \end{bmatrix}$$

$$\therefore \quad |s\boldsymbol{I} - \boldsymbol{A}| = \begin{vmatrix} s & -1 \\ 4 & s+5 \end{vmatrix} = s^2 + 5s + 4$$

$$\therefore \quad (s\boldsymbol{I} - \boldsymbol{A})^{-1} = \frac{adj\ (s\boldsymbol{I} - \boldsymbol{A})}{|s\boldsymbol{I} - \boldsymbol{A}|}$$

$$= \frac{1}{s^2 + 5s + 4} \begin{bmatrix} s+5 & 1 \\ -4 & s \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+5}{s^2+5s+4} & \dfrac{1}{s^2+5s+4} \\ \dfrac{-4}{s^2+5s+4} & \dfrac{s}{s^2+5s+4} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{4}{3(s+1)} - \dfrac{1}{3(s+4)} & \dfrac{1}{3(s+1)} - \dfrac{1}{3(s+4)} \\ \dfrac{-4}{3(s+1)} + \dfrac{4}{3(s+4)} & \dfrac{-1}{3(s+1)} + \dfrac{4}{3(s+4)} \end{bmatrix}$$

$$\Phi(t) = e^{At} = L^{-1}\left[(s\boldsymbol{I} - \boldsymbol{A})^{-1} \right]$$

$$= \begin{bmatrix} \left(\dfrac{4}{3}e^{-t} - \dfrac{1}{3}e^{-4t} \right) & \left(\dfrac{1}{3}e^{-t} - \dfrac{1}{3}e^{-4t} \right) \\ \left(-\dfrac{4}{3}e^{-t} + \dfrac{4}{3}e^{-4t} \right) & \left(-\dfrac{1}{3}e^{-t} + \dfrac{4}{3}e^{-4t} \right) \end{bmatrix}$$

(b) Zero input response is obtained from by putting $u(\tau) = 0$ and letting it be $\boldsymbol{x}_1$.

$$\therefore \quad \boldsymbol{x}_1 = e^{At}\boldsymbol{x}(0) = \boldsymbol{\Phi}(t)\, x(0)$$

$$= \begin{bmatrix} \frac{1}{3}(4e^{-t} - e^{4t}) & \frac{1}{3}(e^{-t} - e^{4t}) \\ \frac{4}{3}(-e^{-t} + e^{-4t}) & \frac{1}{3}(-e^{-t} + 4e^{-4t}) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}(5e^{-t} - 2e^{-4t}) \\ \frac{1}{3}(-5e^{-t} + 8e^{-4t}) \end{bmatrix}$$

(c) Zero state response for $u = 1$ is obtained from by putting $\boldsymbol{x}(0)$ and letting it be $\boldsymbol{x}_2$.

$$\therefore \quad \boldsymbol{x}_2 = \int_o^t e^{A(t-\tau)}\boldsymbol{B}u(\tau)d\tau$$

$$= \int_0^t \begin{bmatrix} \frac{1}{3}\left(4e^{-(t-\tau)} - e^{-4(t-\tau)}\right) & \frac{1}{3}\left(e^{-(t-\tau)} - e^{-4(t-\tau)}\right) \\ \frac{4}{3}\left(-e^{-(t-\tau)} + e^{-4(t-\tau)}\right) & \frac{1}{3}\left(-e^{-(t-\tau)} + 4e^{-4(t-\tau)}\right) \end{bmatrix} \boldsymbol{B}d\tau$$

$$= \int_0^t \begin{bmatrix} \frac{1}{3}\left(e^{-(t-\tau)} - e^{-4(t-\tau)}\right) \\ \frac{1}{3}\left(-e^{-(t-\tau)} + 4e^{-4(t-\tau)}\right) \end{bmatrix} d\tau$$

$$= \begin{bmatrix} \int_0^t \frac{1}{3}\left(e^{-t+\tau} - e^{-4t+4\tau}\right) d\tau \\ \int_0^t \frac{1}{3}\left(-e^{-t+\tau} + 4e^{-4t+4\tau}\right) d\tau \end{bmatrix} = \begin{bmatrix} \left(1 - \frac{1}{4}\right) - \left(e^{-t} - \frac{1}{4}e^{-4t}\right) \\ (-1+1) - \left(-e^{-t} + e^{-4t}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4} - e^{-t} + \frac{1}{4}e^{-4t} \\ e^{-t} - e^{-4t} \end{bmatrix}$$

(d) Total response is given by

$$\boldsymbol{x}(t) = \boldsymbol{x}_1 + \boldsymbol{x}_2$$

$$= \begin{bmatrix} \frac{1}{3}\left(5e^{-t} - 2e^{-4t}\right) \\ \frac{1}{3}\left(-5e^{-t} + 8e^{-4t}\right) \end{bmatrix} + \begin{bmatrix} \frac{3}{4} - e^{-t} + \frac{1}{4}e^{-4t} \\ e^{-t} - e^{-4t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4} + \frac{2}{3}e^{-t} - \frac{5}{12}e^{-4t} \\ -\frac{2}{3}e^{-t} + \frac{5}{3}e^{-4t} \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{3}{4} + \frac{2}{3}e^{-t} - \frac{5}{12}e^{-4t} \\ -\frac{2}{3}e^{-t} + \frac{5}{3}e^{-4t} \end{bmatrix}$$

(e) Inverse of $\Phi(t)$ as shown below.

$$\Phi^{-1}(t) = e^{-At} = \Phi(-t)$$

$$= \begin{bmatrix} \frac{1}{3}\left(4e^{t} - e^{4t}\right) & \frac{1}{3}\left(e^{t} - e^{4t}\right) \\ \frac{4}{3}\left(e^{4t} - e^{t}\right) & \frac{1}{3}\left(4e^{4t} - e^{t}\right) \end{bmatrix}$$

CONCEPTS OF CONTROLLABILITY AND OBSERVABILITY [KALMAN'S TEST]

CONTROLLABILITY

A linear Time invariant system can be represented by $\dot{x} = Ax + Bu$

$y = Cx + Du$ is said to be completely controllable if there exists an input vector $u(t)$ which transfer the system from initial state $x(t_0)$ to the state $x(t_f)$ in a finite time.

For controllability of the system,

$$Q_c = [B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, \ldots\ldots \,\vdots\, A^{n-1}B]$$

where n is rank.

If $|Q_c| = 0$ then the system is not controllable.

If $|Q_c| \neq 0$ then the system is controllable.

OBSERVABILITY

A linear time invariant system can be represented by $\dot{x} = Ax + Bu$

$y = Cx + Du$ is completely observable if the knowledge of the o/ps y and i/ps u over a finite interval $t_0 \leq t \leq t_f$ sufficient to determine every state $x(t_0)$.

For observability of the system,

$$Q_O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

If $|Q_O| = 0$, then the system is not observable.

If $|Q_O| \neq 0$, then the system is observable.

PROBLEMS

1) Find the controllability of the system described by the state equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

SOL:

Given $A = \begin{bmatrix} -2 & 4 \\ 2 & -1 \end{bmatrix}$ & $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore n = 2$

The controllable matrix $Q_C = [B \quad AB]$

$$AB = \begin{bmatrix} -2 & 4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\therefore Q_C = \begin{bmatrix} 0 & 4 \\ 1 & -1 \end{bmatrix}$$

$$|Q_C| = -4$$

$\therefore$ The given system is controllable.

2) Find the observability of the system described by the state equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

SOL:

Given $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \end{bmatrix}$

$\therefore n = 2$

The observable matrix, $Q_o = \begin{bmatrix} C \\ CA \end{bmatrix}$

$$CA = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -4 \end{bmatrix}$$

$$\therefore Q_0 = \begin{bmatrix} 1 & 2 \\ -1 & -4 \end{bmatrix}$$

$$|Q_0| = -2$$

$\therefore$ The given system is observable.

3) Check the controllability of the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ -2 \end{bmatrix} u$$

SOL:

Given $B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$AB = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 \\ -2 \end{bmatrix}_{2\times 1} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 1 & -2 \\ -2 & -4 \end{bmatrix}$$

$$|Q_c| = \begin{vmatrix} 1 & -2 \\ -2 & -4 \end{vmatrix} = -4 - 4 = -8 \neq 0.$$

The system is controllable.

4) Check the observability of the system

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$Y(t) = \begin{bmatrix} 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

SOL:

$$C^T = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} \qquad A^T C^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -7 \\ -1 \end{bmatrix}$$

$$(A^T)^2 C^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} 6 \\ -7 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \\ -1 \end{bmatrix}$$

The rank of of the observability matrix

$$Q_o = \left[C^T \quad A^T C^T \quad (A^T)^2 C^T \right] = \begin{bmatrix} 4 & -6 & -6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{bmatrix}$$

$$|Q_o| = \begin{vmatrix} 4 & -6 & -6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{vmatrix} = 4[7+5] + 6[-5-5] - 6[-5+7]$$

$$= 48 - 60 - 12$$

$$= -24 \neq 0.$$

$\therefore$ the system is completely Observable.

5) Test the observability of the system described by

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

SOL:

Given the matrices A & C

$$C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad A^T C^T = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

The Observability matrix

$$Q_o = \left[C^T \quad A^T C^T \right] = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$|Q_o| = 0 - 0 = 0.$$

Hence the system is not observable

6)

Consider the system defined by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad C = [10 \quad 5 \quad 1]$$

Check the system for (a) complete state controllability and (b) complete observability.

SOL:

(a) Test for complete state controllability

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -12 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -12 \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \\ 61 \end{bmatrix}$$

So the controllability matrix Q_c is given by

$$Q_c = \left[B \,:\, AB \,:\, A^2B \right]$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -12 \\ 1 & -12 & 61 \end{bmatrix}$$

Now, $|Q_c| = -84 \neq 0$

So the rank of matrix Q_c is equal to its order, that is, 3. This indicates that according to Kalman's test, the system is completely state controllable.

(b) Test for complete observability

$$C^* = \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix} \text{ and } A^* = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}$$

$$A^*C^* = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -1 \\ -1 \end{bmatrix}$$

$$(A^*)^2C^* = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} -6 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 5 \end{bmatrix}$$

So the observability matrix Q_0 is given by

$$Q_0 = \left[C^* : A^*C^* : (A^*)^2C^*\right]$$

$$= \begin{bmatrix} 10 & -6 & 6 \\ 5 & -1 & 5 \\ 1 & -1 & 5 \end{bmatrix}$$

Now, $|Q_0| = 96 \neq 0$

So, the rank of matrix Q_0 is equal to its order, that is, 3. This indicates that due to Kalman the system is completely observable.